CS70: Lecture 9. Outline.

1. Public Key Cryptography
2. RSA system
   2.1 Efficiency: Repeated Squaring.
   2.2 Correctness: Fermat's Theorem.
   2.3 Construction.
3. Warnings.

Isomorphisms.

Bijection:

$f(x) = ax \pmod{m}$ if $gcd(a,m) = 1$.

Simplified Chinese Remainder Theorem:
If $gcd(n,m) = 1$, there is unique $x \pmod{mn}$ where
$x = a \pmod{m}$ and $x = b \pmod{n}$.

Bijection between $(a \pmod{n}, b \pmod{m})$ and $x \pmod{mn}$.

Consider $m = 5$, $n = 9$, then if $(a,b) = (3,7)$ then $x = 43 \pmod{45}$.

Consider $(a',b') = (2,4)$, then $x = 22 \pmod{45}$.

Now consider: $(a,b) + (a',b') = (0,2)$.

What is x where $x = 0 \pmod{5}$ and $x = 2 \pmod{9}$?

Try $43 + 22 = 65 = 20 \pmod{45}$.

Is it $0 \pmod{5}$? Yes! Is it $2 \pmod{9}$? Yes!

Isomorphism:
the actions under $\pmod{5}$, $\pmod{9}$
correspond to actions in $\pmod{45}$!

Poll

$x = 5 \mod 7$ **and** $x = 5 \mod 6$
$y = 4 \mod 7$ **and** $y = 3 \mod 6$

What's true?

(A) $x + y = 2 \mod 7$
(B) $x + y = 2 \mod 6$
(C) $xy = 3 \mod 6$
(D) $xy = 6 \mod 7$
(E) $x = 5 \mod 42$
(F) $y = 39 \mod 42$

All true.

Xor

Computer Science:
 1 - True
 0 - False

$1 \vee 1 = 1$
$1 \vee 0 = 1$
$0 \vee 1 = 1$
$0 \vee 0 = 0$

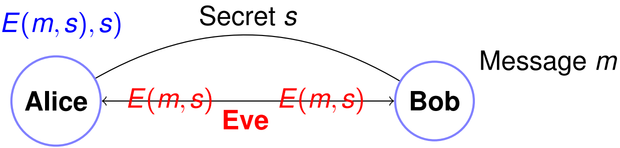
$A \oplus B$ - Exclusive or.
$1 \oplus 1 = 0$
$1 \oplus 0 = 1$
$0 \oplus 1 = 1$
$0 \oplus 0 = 0$

Note: Also modular addition modulo 2!
 $\{0, 1\}$ is set. Take remainder for 2.

Property: $A \oplus B \oplus B = A$.
By cases: $1 \oplus 1 \oplus 1 = 1$. $\ldots$

Cryptography ...

$m = D(E(m,s),s)$



Example:

One-time Pad: secret s is string of length $|m|$.

$m = 1010101110101101$

$s \;= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$E(m,s)$ – bitwise $m \oplus s$.

$D(x,s)$ – bitwise $x \oplus s$.

Works because $m \oplus s \oplus s = m$!

...and totally secure!

...given $E(m,s)$ any message m is equally likely.

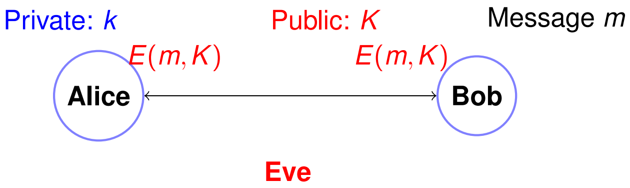
Disadvantages:

Shared secret!

Uses up one time pad..or less and less secure.

Public key crypography.

$m = D(E(m,K),k)$



Everyone knows key K!
Bob (and Eve and me and you and you ...) can encode.
Only Alice knows the secret key k for public key K.
(Only?) Alice can decode with k.

Is this even possible?

Is public key crypto possible?

No. In a sense. One can try every message to "break" system. Too slow. Does it have to be slow? We don't really know.
...but we do public-key cryptography constantly!!!

RSA (Rivest, Shamir, and Adleman)
Pick two large primes p and q. Let $N = pq$.
Choose e relatively prime to $(p-1)(q-1)$.[1]
Compute $d = e^{-1} \mod (p-1)(q-1)$.
Announce $N (= p \cdot q)$ and e: $K = (N, e)$ is my public key!

Encoding: $\mod (x^e, N)$.

Decoding: $\mod (y^d, N)$.

Does $D(E(m)) = m^{ed} = m \mod N$?

Yes!

[1] Typically small, say $e = 3$.

Poll

What is a piece of RSA?

Bob has a key (N,e,d). Alice is good, Eve is evil.

(A) Eve knows e and N.
(B) Alice knows e and N.
(C) $ed = 1 \pmod{N-1}$
(D) Bob forgot p and q but can still decode?
(E) Bob knows d
(F) $ed = 1 \pmod{(p-1)(q-1)}$ if $N = pq$.

(A), (B), (D), (E), (F)

Iterative Extended GCD.

Example: $p = 7$, $q = 11$.

$N = 77$.
$(p-1)(q-1) = 60$
Choose $e = 7$, since $\gcd(7,60) = 1$.
egcd(7,60).

$$\begin{aligned}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3 \\
7(-17) + 60(2) &= 1
\end{aligned}$$

Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

Encryption/Decryption Techniques.

Public Key: $(77, 7)$
Message Choices: $\{0, \ldots, 76\}$.

Message: 2!

$E(2) = 2^e = 2^7 \equiv 128 = 51 \pmod{77}$
$D(51) = 51^{43} \pmod{77}$
uh oh!

Obvious way: 43 multiplications. Ouch.

In general, $O(N)$ or $O(2^n)$ multiplications!

Repeated squaring.

Notice: $43 = 32+8+2+1$ or 101011 in binary.
$51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.
4 multiplications sort of...
Need to compute $51^{32} \ldots 51^{1}$.?
$51^{1} \equiv 51 \pmod{77}$
$51^{2} = (51)*(51) = 2601 \equiv 60 \pmod{77}$
$51^{4} = (51^{2})*(51^{2}) = 60*60 = 3600 \equiv 58 \pmod{77}$
$51^{8} = (51^{4})*(51^{4}) = 58*58 = 3364 \equiv 53 \pmod{77}$
$51^{16} = (51^{8})*(51^{8}) = 53*53 = 2809 \equiv 37 \pmod{77}$
$51^{32} = (51^{16})*(51^{16}) = 37*37 = 1369 \equiv 60 \pmod{77}$

5 more multiplications.

$51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} = (60)*(53)*(60)*(51) \equiv 2 \pmod{77}$.

Decoding got the message back!

Repeated Squaring took 9 multiplications versus 43.

Recursive version.

```
(define (power x y m)
  (if (= y 1)
    (mod x m)
    (let ((x-to-evened-y  (power (square x) (/ y 2) m)))
    (if (evenp y)
        x-to-evened-y
        (mod (* x x-to-evened-y) m )))))
```

Claim: Program correctly computes x^y.

Base: $x^1 = x \pmod{m}$.

Note: $y = 2\lfloor y/2 \rfloor + \mod(y,2)$.

$x^y = x^{2(\lfloor y/2 \rfloor) + \mod(y,2)} = (x^2)^{\lfloor y/2 \rfloor} x^{y \mod 2} \pmod{m}$.

Induction:

Recursive call on x^2 and $\lfloor y/2 \rfloor$, returns $(x^2)^{\lfloor y/2 \rfloor}$.

≤ 2 multiplications per recursive call.

Note: $\lfloor y/2 \rfloor$ is integer division.

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.
2. Multiply together x^i where the $(\log(i))$th bit of y (in binary) is 1.
   Example: $43 = 101011$ in binary.
   $$x^{43} = x^{32} * x^8 * x^2 * x^1.$$

Modular Exponentiation: $x^y \mod N$. All n-bit numbers. Repeated Squaring:

$O(n)$ multiplications.
$O(n^2)$ time per multiplication.
$\implies O(n^3)$ time.

Conclusion: $x^y \mod N$ takes $O(n^3)$ time.

RSA is pretty fast.

Modular Exponentiation: $x^y \mod N$. All n-bit numbers.
$O(n^3)$ time.

Remember RSA encoding/decoding!

$E(m,(N,e)) = m^e \pmod{N}$.
$D(m,(N,d)) = m^d \pmod{N}$.

For 512 bits, a few hundred million operations.
Easy, peasey.

Decoding.

$$E(m,(N,e)) = m^e \pmod{N}.$$
$$D(m,(N,d)) = m^d \pmod{N}.$$

$N = pq$ and $d = e^{-1} \pmod{(p-1)(q-1)}$.

Want: $(m^e)^d = m^{ed} = m \pmod{N}$.

Always decode correctly?

$E(m,(N,e)) = m^e \pmod N$.
$D(m,(N,d)) = m^d \pmod N$.

$N = pq$ and $d = e^{-1} \pmod{(p-1)(q-1)}$.

Want: $(m^e)^d = m^{ed} = m \pmod N$.

Another view:
$d = e^{-1} \pmod{(p-1)(q-1)} \iff ed = k(p-1)(q-1)+1$.

Consider...

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod p$,

$$a^{p-1} \equiv 1 \pmod p.$$

$$\implies a^{k(p-1)} \equiv 1 \pmod p \implies a^{k(p-1)+1} = a \pmod p$$

versus $\quad a^{k(p-1)(q-1)+1} = a \pmod{pq}$.

Similar, not same, but useful.

Correct decoding...

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod{p}$,

$a^{p-1} \equiv 1 \pmod{p}$.

Proof: Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo p since a has an inverse modulo p.
S contains representative of $\{1, \ldots, p-1\}$ modulo p.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \mod p.$$

Each of $2, \ldots (p-1)$ has an inverse modulo p, solve to get...

$$a^{(p-1)} \equiv 1 \mod p.$$

$\square$

Poll

Mark what is true.

(A) $2^7 = 1 \mod 7$
(B) $2^6 = 1 \mod 7$
(C) $2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$ are distinct mod 7.
(D) $2^1, 2^2, 2^3, 2^4, 2^5, 2^6$ are distinct mod 7
(E) $2^{15} = 2 \mod 7$
(F) $2^{15} = 1 \mod 7$

(B), (F)

Always decode correctly? (cont.)

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod{p}$,

$$a^{p-1} \equiv 1 \pmod{p}.$$

Lemma 1: For any prime p and any a, b,
$a^{1+b(p-1)} \equiv a \pmod{p}$
Proof: If $a \equiv 0 \pmod{p}$, of course.

Otherwise
$a^{1+b(p-1)} \equiv a^1 * (a^{p-1})^b \equiv a * (1)^b \equiv a \pmod{p}$

$\square$

...Decoding correctness...

Lemma 1: For any prime p and any a, b,

$$a^{1+b(p-1)} \equiv a \pmod{p}$$

Lemma 2: For any two different primes p, q and any x, k,

$$x^{1+k(p-1)(q-1)} \equiv x \pmod{pq}$$

Let $a = x$, $b = k(p-1)$ and apply Lemma 1 with modulus q.

$$x^{1+k(p-1)(q-1)} \equiv x \pmod{q}$$

Let $a = x$, $b = k(q-1)$ and apply Lemma 1 with modulus p.

$x^{1+k(p-1)(q-1)} \equiv x \pmod{p}$ $x^{1+k(q-1)(p-1)} - x$ is multiple of p and q.

$$x^{1+k(q-1)(p-1)} - x \equiv 0 \mod (pq) \implies x^{1+k(q-1)(p-1)} = x \mod pq.$$

$\square$

From CRT: $y = x \pmod{p}$ and $y = x \pmod{q} \implies y = x$.

RSA decodes correctly..

Lemma 2: For any two different primes p, q and any x, k,

$$x^{1+k(p-1)(q-1)} \equiv x \pmod{pq}$$

Theorem: RSA correctly decodes!
Recall

$$D(E(x)) = (x^e)^d = x^{ed} \equiv x \pmod{pq},$$

where $ed \equiv 1 \mod (p-1)(q-1) \implies ed = 1 + k(p-1)(q-1)$

$$x^{ed} \equiv x^{k(p-1)(q-1)+1} \equiv x \pmod{pq}.$$

$\square$

Construction of keys.. ..

1. Find large (100 digit) primes p and q?

   Prime Number Theorem: $\pi(N)$ number of primes less than N.For all $N \geq 17$

   $$\pi(N) \geq N/\ln N.$$

   Choosing randomly gives approximately $1/(\ln N)$ chance of number being a prime. (How do you tell if it is prime? ... cs170..Miller-Rabin test.. Primes in P).

   For 1024 bit number, 1 in 710 is prime.

2. Choose e with $\gcd(e, (p-1)(q-1)) = 1$.
   Use gcd algorithm to test.

3. Find inverse d of e modulo $(p-1)(q-1)$.
   Use extended gcd algorithm.

All steps are polynomial in $O(\log N)$, the number of bits.

Security of RSA.

Security?

1. Alice knows p and q.
2. Bob only knows, $N(= pq)$, and e.

   Does not know, for example, d or factorization of N.
3. I don't know how to break this scheme without factoring N.

No one I know or have heard of admits to knowing how to factor N.
Breaking in general sense $\implies$ factoring algorithm.

Much more to it.....

If Bobs sends a message (Credit Card Number) to Alice,

Eve sees it.

Eve can send credit card again!!

The protocols are built on RSA but more complicated;

For example, several rounds of challenge/response.

One trick:
  Bob encodes credit card number, c,
    concatenated with random k-bit number r.

Never sends just c.

Again, more work to do to get entire system.

CS161...

Signatures using RSA.

Verisign: k_v, K_v

$[C, S_v(C)]$ $\qquad$ $C = E(S_V(C), k_V)?$

$[C, S_v(C)]$ $\qquad$ $[C, S_v(C)]$

Amazon $\longleftrightarrow$ **Browser.** K_v

Certificate Authority: Verisign, GoDaddy, DigiNotar,...

Verisign's key: $K_V = (N, e)$ and $k_V = d$ $(N = pq.)$

Browser "knows" Verisign's public key: K_V.

Amazon Certificate: C = "I am Amazon. My public Key is K_A."

Versign signature of C: $S_v(C)$: $D(C, k_V) = C^d \mod N$.

Browser receives: $[C, y]$

Checks $E(y, K_V) = C$?

$E(S_v(C), K_V) = (S_v(C))^e = (C^d)^e = C^{de} = C \pmod N$

Valid signature of Amazon certificate C!

Security: Eve can't forge unless she "breaks" RSA scheme.

RSA

Public Key Cryptography:

$D(E(m, K), k) = (m^e)^d \mod N = m.$

Signature scheme:

$E(D(C, k), K) = (C^d)^e \mod N = C$

Poll

Signature authority has public key (N,e).

(A) Given message/signature (x,y) : check $y^d = x \pmod N$
(B) Given message/signature (x, y): check $y^e = x \pmod N$
(C) Signature of message x is $x^e \pmod N$
(D) Signature of message x is $x^d \pmod N$

Other Eve.

Get CA to certify fake certificates: Microsoft Corporation.

2001..Doh.

... and August 28, 2011 announcement.

DigiNotar Certificate issued for Microsoft!!!

How does Microsoft get a CA to issue certificate to them ...

and only them?

Summary.

Public-Key Encryption.

RSA Scheme:
$N = pq$ and $d = e^{-1} \pmod{(p-1)(q-1)}$.
$E(x) = x^e \pmod{N}$.
$D(y) = y^d \pmod{N}$.

Repeated Squaring $\implies$ efficiency.

Fermat's Theorem $\implies$ correctness.

Good for Encryption and Signature Schemes.