Lecture 7. Outline.

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4. Euclid's GCD Algorithm.
   A little tricky here!

Isoperimetry.

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Surface Area is roughly at least the volume!

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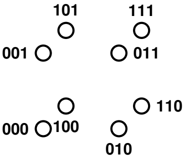

(A) Lower left forward node name is 0000
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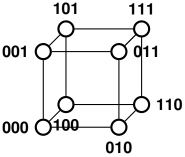

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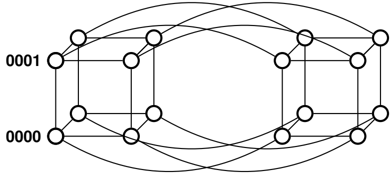

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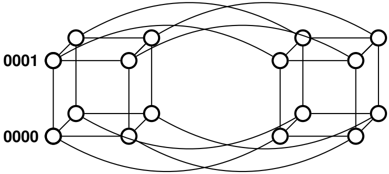

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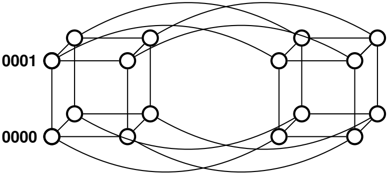

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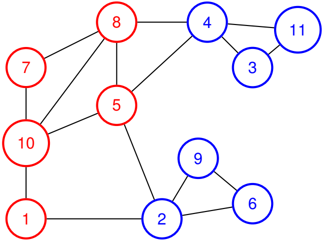
Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Cuts in graphs.



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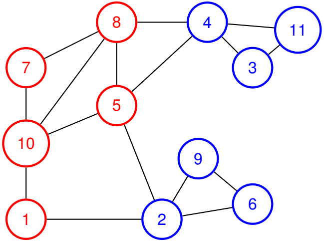

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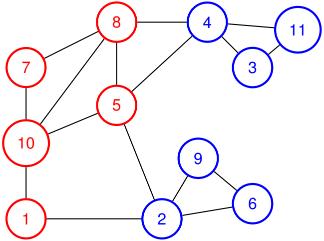

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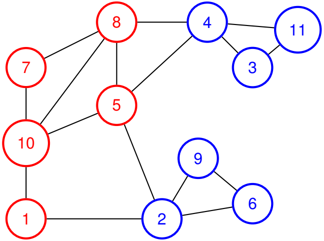

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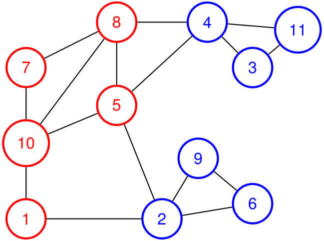

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Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

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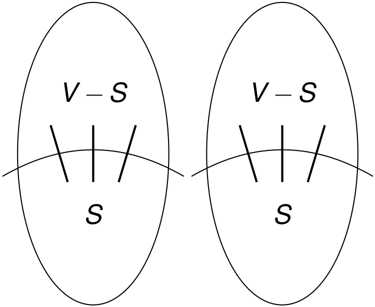

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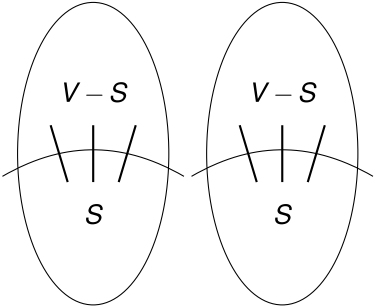

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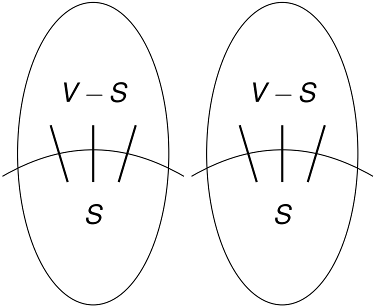

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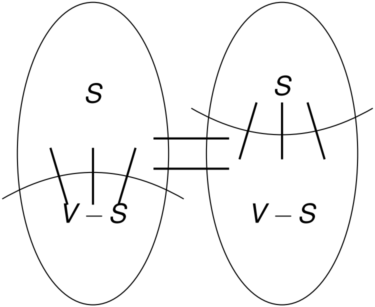

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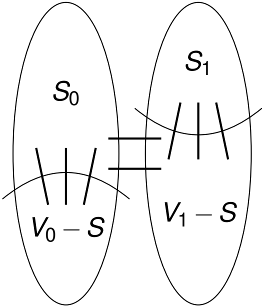
Proof: Induction Step. Case 2.

$$|S_0| \geq |V_0|/2.$$



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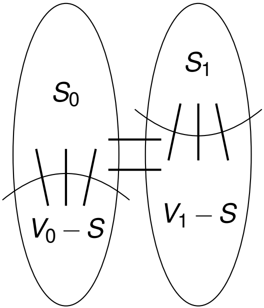

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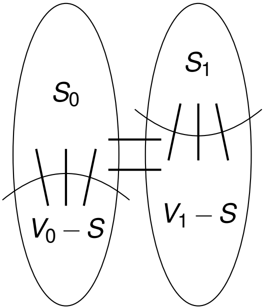

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$\implies \geq |S_1|$ edges cut in E_1.

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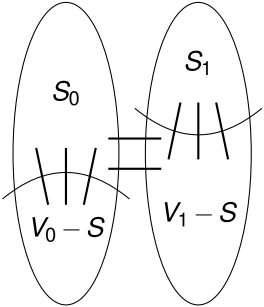

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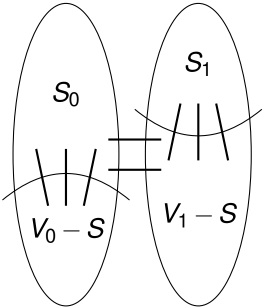

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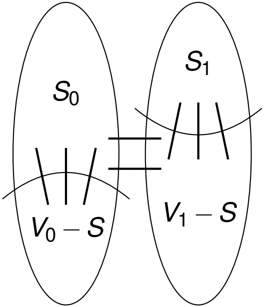

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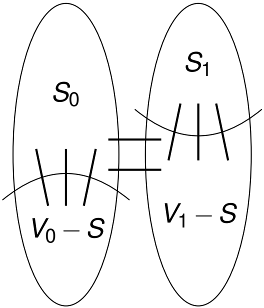

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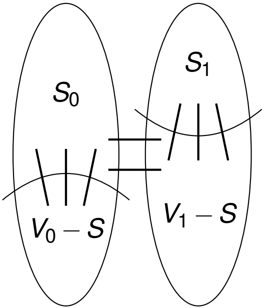

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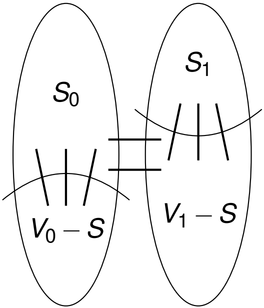

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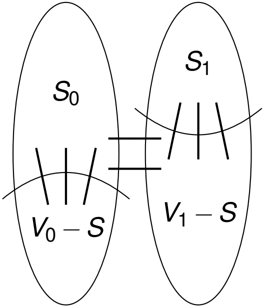

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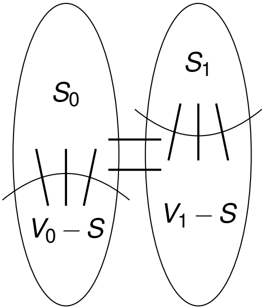

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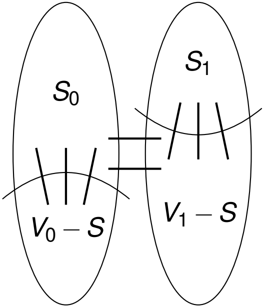

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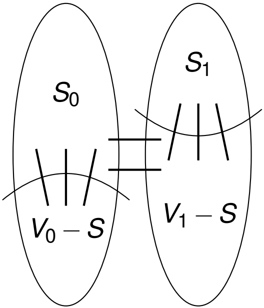

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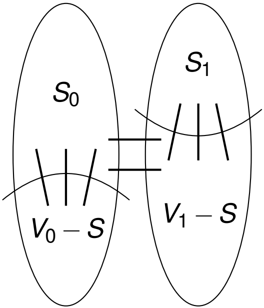

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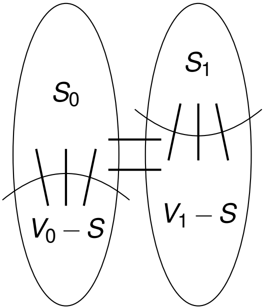

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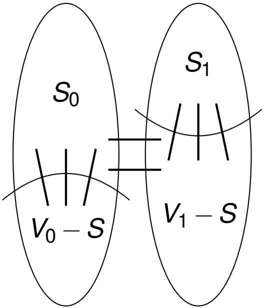

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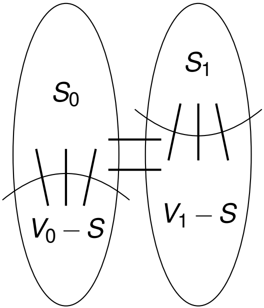

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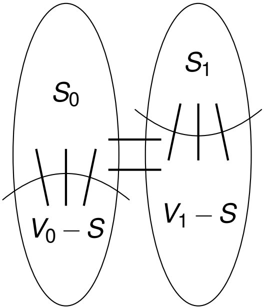

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Hypercube proof: poll

Hypercube has large cuts proof uses these ideas:
(A) If cuts are same size on two sides it works by induction.
(B) Uses the fact that it is planar.
(C) Recursive definition of hypercube.
(D) If different size, can count edges between to subcubes.
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(A),(D), and (E).

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$.

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Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.

Modular Arithmetic.

Applications: cryptography, error correction.

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

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$\square$

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. $\square$
(Uniqueness? Later.)

Poll

What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler's formula.
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(A) and (C)

Next Up.

Modular Arithmetic.

Clock Math

If it is 1:00 now.

Clock Math

If it is 1:00 now.
What time is it in 2 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
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Clock Math

If it is 1:00 now.

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Actually 4:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

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Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
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Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

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If it is 1:00 now.

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Clock Math

If it is 1:00 now.
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  - Actually 4:00.

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What time is it in 100 hours?

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

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What time is it in 100 hours? 101:00!

Clock Math

If it is 1:00 now.

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Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

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If it is 1:00 now.
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Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
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What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.

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What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.

Clock Math

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Actually 4:00.

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What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
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Clock time equivalent up to addition of any integer multiple of 12.

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What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$
(Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

This is Thursday is September 16, 2021.

Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then?

Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

This is Thursday is September 16, 2021.

- What day is it a year from then? on September 16, 2022?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then.

Day of the week.

This is Thursday is September 16, 2021.

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Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9

Day of the week.

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Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then.

Day of the week.

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Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29

Day of the week.

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5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1.

Day of the week.

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Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

Day of the week.

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25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then

Day of the week.

This is Thursday is September 16, 2021.
  What day is it a year from then? on September 16, 2022?
    Number days.
      0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. $29 = (7)4 + 1$
    two days are equivalent up to addition/subtraction of multiple of 7.
    11 days from then is day 1

Day of the week.

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two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

Day of the week.

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What day is it a year from then? on September 16, 2022?

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two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Day of the week.

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What day is it a year from then? on September 16, 2022?

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25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

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Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day of the week.

This is Thursday is September 16, 2021.

- What day is it a year from then? on September 16, 2022?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

- 5 days from then. day 9 or day 2 or Tuesday.
- 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then is day 1 which is Monday!

What day is it a year from then?

- Next year is not a leap year. So 365 days from then.
- Day 4+365 or day 369.

Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:

Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
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25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.

Day of the week.

This is Thursday is September 16, 2021.
 What day is it a year from then? on September 16, 2022?
  Number days.
   0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
 5 days from then. day 9 or day 2 or Tuesday.
 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from then is day 1 which is Monday!

What day is it a year from then?
 Next year is not a leap year. So 365 days from then.
 Day 4+365 or day 369.
Smallest representation:
 subtract 7 until smaller than 7.
 divide and get remainder.

Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7

Day of the week.

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Today: day 4.

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25 days from then. day 29 or day 1. $29 = (7)4 + 1$

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11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

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two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

or September 16, 2022 is a Friday.

Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

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What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

or September 16, 2022 is a Friday.

Years and years...

80 years?

Years and years...

80 years? 20 leap years.

Years and years...

80 years? 20 leap years. 366×20 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60$

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Years and years...

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60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Or February 11, 2101 is Saturday!

Further Simplify Calculation:

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.

Years and years...

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What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
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Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 6.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 6. September 16, 2101 is Saturday.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
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Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $2 + 2 \times 6 + 1 \times 4 = 18$.
Or Day 6. September 16, 2101 is Saturday.

"Reduce" at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or $x = y + km$ for some integer k.

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
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...or $x = y + km$ for some integer k.

Mod 7 equivalence classes:

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or $x = y + km$ for some integer k.

Mod 7 equivalence classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x-y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or $x = y + km$ for some integer k.

Mod 7 equivalence classes:
$\{\ldots,-7,0,7,14,\ldots\}$ $\{\ldots,-6,1,8,15,\ldots\}$

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
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Mod 7 equivalence classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$ $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
...or x and y have the same remainder w.r.t. m.
...or $x = y + km$ for some integer k.

Mod 7 equivalence classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$ $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.
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Mod 7 equivalence classes:
$\{\ldots,-7,0,7,14,\ldots\}$ $\{\ldots,-6,1,8,15,\ldots\}$ ...

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

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Can calculate with representative in $\{0,\ldots,m-1\}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$

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Work in this system.

Notation

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Work in this system.

$a \equiv b \pmod{m}$.

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Work in this system.

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Says two integers a and b are equivalent modulo m.

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Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

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Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}.$$

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Check!

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Check! $4(3) = 12 = 5 \pmod{7}$.

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$x = 3 \pmod{7}$ ::: Check! $4(3) = 12 = 5 \pmod{7}$.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

Multiplicative inverse of x is y where $xy = 1$;
1 **is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

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For 4 modulo 7 inverse is 2: $2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}$.

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Poll

Mark true statements.

(A) Mutliplicative inverse of 2 mod 5 is 3 mod 5.

(B) The multiplicative inverse of $((n-1) \pmod{n}) = ((n-1) \pmod{n})$.

(C) Multiplicative inverse of 2 mod 5 is 0.5.

(D) Multiplicative inverse of $4 = -1 \pmod{5}$.

(E) $(-1)x(-1) = 1$. Woohoo.

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(C) is false. 0.5 has no meaning in arithmetic modulo 5.

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Thm:
If greatest common divisor of x and m, $\gcd(x, m)$, is 1, then x has a multiplicative inverse modulo m.

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For $x = 5$ and $m = 6$.

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All distinct, contains 1!

Proof review. Consequence.

Thm: If $\gcd(x,m) = 1$, then x has a multiplicative inverse modulo m.

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Very different for elements with inverses.

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Not a bijection.

Poll

Which is bijection?

(A) $f(x) = x$ for domain and range being $\mathbb{R}$

(B) $f(x) = ax \pmod{()} n)$ for $x \in \{0, ..., n-1\}$ and $gcd(a, n) = 2$

(C) $f(x) = ax \pmod{n}$ for $x \in \{0, ..., n-1\}$ and $gcd(a, n) = 1$

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(B) is not.

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$a(nd) = 1 + k\ell d$ or $d(na - k\ell) = 1$.

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Finding inverses.

How to find the inverse?

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How to find **if** x has an inverse modulo m?

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Euclid's Algorithm.

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Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer k.

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