CS70.

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Which are parts of proof?

(A) $k^3 - k = qn$ for $q \in \mathbb{N}$.

(B) $0^3 - 0 = 0$, $3|0$ since $3 = 0(3)$.

(C) $(k+1)^3 - (k+1) = k^3 + 2k$.

(D) $k^3 + 2k = k(k^2 + 2)$.

(E) Add $k - k$ to $k^3 + 2k$.

(F) $(k^3 - k) + 3k = 3(q + k)$.

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Add $(k - k)$.

CS70

Induction: Some quibbles.

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What did you learn in 61A?

CS70

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CS70

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Couple of more induction proofs.

CS70

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Induction and Recursion

Couple of more induction proofs.

Stable Marriage.

Some quibbles.

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What if the statement is only for $n \geq 3$?

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Since $\forall n \in \mathbb{N}, Q(n) \implies Q(n+1)$ is trivially true before 3.

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In some sense, the natural numbers.

Poll

Question: What is the Main Idea of 61A

(A) Functional Programming.

(B) Environment Diagrams.

(C) Recursion.

(D) John Denero is kind of dreamy.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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Base cases: P(12)

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Prove: Given n, returns (x, y) where $n = 4x + 5y$, for $n \geq 12$.

Base cases: P(12) , P(13)

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Recursive call is correct: $P(n-4)$

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Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Strengthening: need to...

Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Induction step works! No!

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Theorem: For all $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$. ($S_n = \sum_{i=1}^{n} \frac{1}{i^2}$.)

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Hmmm... It better be that any sum is *strictly less than* 2.

How much less? At least by $\frac{1}{(k+1)^2}$ for S_k.

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Maybe a linearly decreasing function to keep positive?

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Stable Matching Problem

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- n candidates and n jobs.

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- Each job has a ranked preference list of candidates.

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How should they be matched?

Count the ways..

- Maximize total satisfaction.

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Count the ways..

- Maximize total satisfaction.
- Maximize number of first choices.
- Maximize worse off.
- Minimize difference between preference ranks.

The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

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Davis prefers the Lakers.

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Davis prefers the Lakers.

Lakers prefer Davis.

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Uh..oh.

The best laid plans..

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Lakers prefer Davis.

Uh..oh. Sad Lonzo and Pelicans.

So..

Produce a matching where there are no crazy moves!

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Definition: A **matching** is disjoint set of n job-candidate pairs.

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Example: A matching $S = \{(Lakers, Ball); (Pelicans, Davis)\}$.

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Definition: A **rogue couple** b, g^* for a pairing S:
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Definition: A **rogue couple** b, g^* for a pairing S:
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Example: Davis and Lakers are a rogue couple in S.

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Given a set of preferences.

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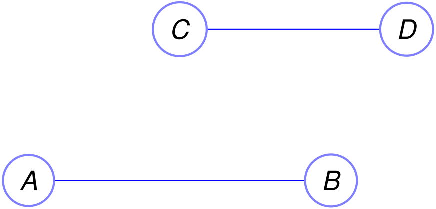
Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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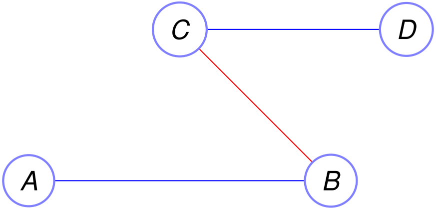
B	C	A	D
C	A	B	D
D	A	B	C



A stable matching??

Given a set of preferences.

Is there a stable matching?
How does one find it?

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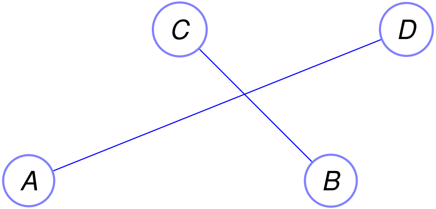

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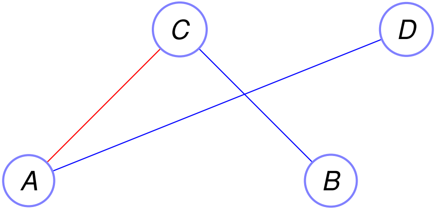

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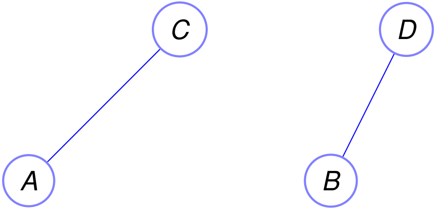

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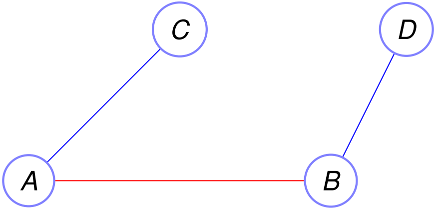

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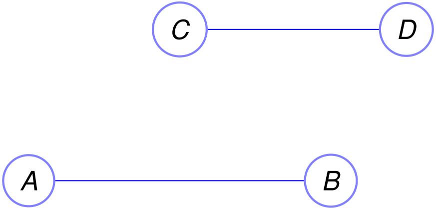

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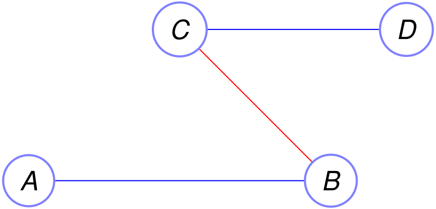

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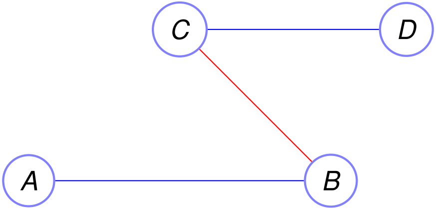

The Propose and Reject Algorithm.

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...produce a matching?

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Do jobs or candidates do “better”?

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~	2	3
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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

Example.

Jobs			
A	1	2	3
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C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, ~~C~~			
3					

Example.

Jobs			
A	1	2	3
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Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	A , C		
2	C	B, ~~C~~	B		
3					

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A	~~1~~	2	3
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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~, C		
2	C	B, ~~C~~	B		
3					

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	2	3
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1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A	~~A~~ , C	C	
2	C	B, ~~C~~	B	A,B	
3					

Example.

Jobs			
A	~~1~~	2	3
B	~~1~~	~~2~~	3
C	~~2~~	1	3

Candidates			
1	C	A	B
2	A	B	C
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1	A, ~~B~~	A	~~A~~ , C	C	
2	C	B, ~~C~~	B	A,~~B~~	
3					

Example.

Jobs			
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3					B

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Jobs			
A	~~1~~	2	3
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	Day 1	Day 2	Day 3	Day 4	Day 5
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3					B

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Terminates in $\leq n^2$ steps!

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Improvement Lemma: It just gets better for candidates

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On day $t+k+1$, job b' comes back.

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That is,

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That is, $b' \leq b$ by induction hypothesis.

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$P(k)$- - "job on g's string is at least as good as b on day $t+k$"

$P(0)$– true. Candidate has b on string.

Assume $P(k)$. Let b' be job **on string** on day $t+k$.

On day $t+k+1$, job b' comes back.
Candidate g can choose b', or do better with another job, b''

That is, $b' \leq b$ by induction hypothesis.
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Improvement Lemma: It just gets better for candidates.

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Poll

Question: It just gets better for candidates, because?

(A) Induction on days.

(B) When the economy is good.

(C) The candidate can always keep the job on the string.

Matching when done.

Lemma: Every job is matched at end.

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Contradiction.

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If not, a job *b* must have been rejected *n* times.

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Contradiction. $\square$

Question: The argument for termination uses.

(A) Implies: no unmatched job at end.

(B) Improvement Lemma: every candidate matched.

(C) Algorithm: unmatched job would ask everyone.

(D) Implies: every one gets their favorite job.

Matching is Stable.

Lemma: There is no rogue couple for the matching formed by traditional marriage algorithm.

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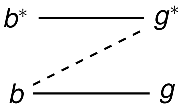

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$$\begin{tikzcd}
b^* \arrow[r, dash] & g^* \\
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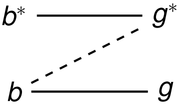

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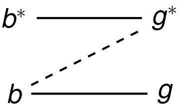

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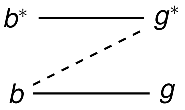

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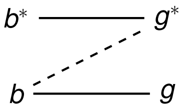

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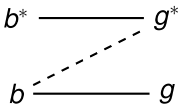

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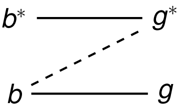

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Good for jobs? candidates?

Is the Job-Proposes better for jobs?

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Yes? No?

Question: The SMA produces a stable pairing is a proof by?

(A) Contradiction.

(B) Uses the impropment lemma.

(C) Induction.

(D) Direct.

Understanding Optimality: by example.

A:	1,2	1:	A,B
B:	1,2	2:	B,A

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B:	1,2	2:	B,A

Consider pairing: $(A,1),(B,2)$.

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

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Pairing T: $(A, 2), (B, 1)$. Also Stable.

Which is optimal for A?

Understanding Optimality: by example.

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Which is optimal for 1?

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Job Propose and Candidate Reject is optimal!

For jobs?

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For jobs? For candidates?

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Theorem: Job Propose and Reject produces a job-optimal pairing.

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Structural statement: Job optimality

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Notes: Not really induction.
Structural statement: Job optimality $\implies$ Candidate pessimality.

Quick Questions.

How does one make it better for candidates?

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Propose and Reject - stable matching algorithm. One side proposes.

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Candidates propose.

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Residency Matching..

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The method was used to match residents to hospitals.

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Hospital optimal....

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Another variation: couples.

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Takeaways.

Analysis of cool algorithm with interesting goal: stability.

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Proofs carefully use definition:

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Proofs carefully use definition:
 Optimality proof:
  contradiction of the existence of a better pairing.