70: Discrete Math and Probability Theory

Programming + Microprocessors $\equiv$ Superpower!

What are your super powerful programs/processors doing?
Logic and Proofs!
Induction $\equiv$ Recursion.

What can computers do?
Work with discrete objects.
Discrete Math $\Longrightarrow$ immense application.

Computers learn and interact with the world?
E.g. machine learning, data analysis, robotics, ...
Probability!

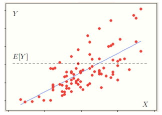
My hopes and dreams.

We teach you to think more clearly and more powerfully.

..And to deal clearly with uncertainty itself.

Probability Unit

- How can we predict unknown future events (e.g., gambling profit, next week rainfall, traffic congestion, ...)?
  - Constructive Models: Model the overall system (including the sources of uncertainty).
    - For modeling uncertainty, we'll develop probabilistic models and techniques for analyzing them.
  - Deductive Models: Extract the "trend" from the previous outcomes (e.g., linear regression).



Admin

Course Webpage: **http://www.eecs70.org/**

Explains policies, has office hours, homework, midterm dates, etc.

One midterm, final.
midterm.

Questions $\Longrightarrow$ piazza:
Logistics, etc.
Content Support: other students!
Plus Piazza hours.

Weekly Post.

It's weekly.
Read it!!!!
Announcements, logistics, critical advice.

Wason's experiment:1

Suppose we have four cards on a table:

- 1st about Alice, 2nd about Bob, 3rd Charlie, 4th Donna.
- Card contains person's destination on one side, and mode of travel.
- Consider the theory: "If a person travels to Chicago, they flies."
- Suppose you see that Alice went to Baltimore, Bob drove, Charlie went to Chicago, and Donna flew.

Alice	Bob	Charlie	Donna
Baltimore	drove	Chicago	flew

- Which cards must you flip to test the theory?

Answer: Later.

CS70: Lecture 1. Outline.

Today: Note 1. Note 0 is background. Do read it.

The language of proofs!

1. Propositions.
2. Propositional Forms.
3. Implication.
4. Truth Tables
5. Quantifiers
6. More De Morgan's Laws

Propositions: Statements that are true or false.

$\sqrt{2}$ is irrational	**Proposition**	True
2+2 = 4	**Proposition**	True
2+2 = 3	**Proposition**	False
826th digit of pi is 4	**Proposition**	False
Johnny Depp is a good actor	**Not Proposition**	
Any even $>$ 2 is sum of 2 primes	**Proposition**	False
$4+5$	**Not Proposition.**	
$x+x$	**Not a Proposition.**	
Alice travelled to Chicago	**Proposition.**	False
I love you.	**Hmmm.**	Its complicated.

Again: "value" of a proposition is ... True or False

Propositional Forms.

Put propositions together to make another...

Conjunction ("and"): $P \wedge Q$

"$P \wedge Q$" is True when both P and Q are True . Else False .

Disjunction ("or"): $P \vee Q$

"$P \vee Q$" is True when at least one P or Q is True . Else False .

Negation ("not"): $\neg P$

"$\neg P$" is True when P is False . Else False .

Examples:

$\neg$ "$(2+2=4)$" – a proposition that is ... False

"$2+2=3$" $\wedge$ "$2+2=4$" – a proposition that is ... False

"$2+2=3$" $\vee$ "$2+2=4$" – a proposition that is ... True

Put them together..

Propositions:
P_1 - Person 1 rides the bus.
P_2 - Person 2 rides the bus.
....

But we can't have either of the following happen; That either person 1 or person 2 ride the bus and person 3 or 4 ride the bus. Or that person 2 or person 3 ride the bus and that either person 4 rides the bus or person 5 doesn't.

Propositional Form:
$\neg(((P_1 \vee P_2) \wedge (P_3 \vee P_4)) \vee ((P_2 \vee P_3) \wedge (P_4 \vee \neg P_5)))$

Can person 3 ride the bus?
Can person 3 and person 4 ride the bus together?

This seems ...complicated.

We can program!!!!

We need a way to keep track!

Truth Tables for Propositional Forms.

"$P \wedge Q$" is True when both P and Q are True .

"$P \vee Q$" is True when $\geq$ one of P or Q is True .

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Check: $\wedge$ and $\vee$ are commutative.

One use for truth tables: Logical Equivalence of propositional forms!

Example: $\neg(P \wedge Q)$ logically equivalent to $\neg P \vee \neg Q$. Same Truth Table!

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

DeMorgan's Law's for Negation: distribute and flip!

$\neg(P \wedge Q) \quad \equiv \quad \neg P \vee \neg Q \qquad\qquad \neg(P \vee Q) \quad \equiv \quad \neg P \wedge \neg Q$

Quick Questions

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Is $(T \wedge Q) \equiv Q$? Yes? No?

Yes! Look at rows in truth table for $P = T$.

What is $(F \wedge Q)$? F or False.

What is $(T \vee Q)$? T

What is $(F \vee Q)$? Q

Distributive?

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$?

Simplify: $(T \wedge Q) \equiv Q$, $(F \wedge Q) \equiv F$.

Cases:
- P is True .
  - LHS: $T \wedge (Q \vee R) \equiv (Q \vee R)$.
  - RHS: $(T \wedge Q) \vee (T \wedge R) \equiv (Q \vee R)$.
- P is False .
  - LHS: $F \wedge (Q \vee R) \equiv F$.
  - RHS: $(F \wedge Q) \vee (F \wedge R) \equiv (F \vee F) \equiv F$.

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$?

Simplify: $T \vee Q \equiv T$, $F \vee Q \equiv Q$. ...

Foil 1:
$(A \vee B) \wedge (C \vee D) \equiv (A \wedge C) \vee (A \wedge D) \vee (B \wedge C) \vee (B \wedge D)$?

Foil 2:
$(A \wedge B) \vee (C \wedge D) \equiv (A \vee C) \wedge (A \vee D) \wedge (B \vee C) \wedge (B \vee D)$?

Implication.

$P \implies Q$ interpreted as

If P, then Q.

True Statements: P, $P \implies Q$.
Conclude: Q is true.

Examples:

Statement: If you stand in the rain, then you'll get wet.
P = "you stand in the rain"
Q = "you will get wet"
Statement: "Stand in the rain"
Can conclude: "you'll get wet."

Statement:
If a right triangle has sidelengths $a \le b \le c$, then $a^2 + b^2 = c^2$.

P = "a right triangle has sidelengths $a \le b \le c$",
Q = "$a^2 + b^2 = c^2$".

Non-Consequences/consequences of Implication

The statement "$P \implies Q$"

only is False if P is True and Q is False .

False implies nothing
P False means Q can be True or False
Anything implies true.
P can be True or False when Q is True

If chemical plant pollutes river, fish die.
If fish die, did chemical plant pollute river?

Not necessarily.

$P \implies Q$ and Q are True does not mean P is True

Be careful!

Instead we have:
$P \implies Q$ and P are True *does* mean Q is True .

The chemical plant pollutes river. Can we conclude fish die?

Some Fun: use propositional formulas to describe implication?
$((P \implies Q) \wedge P) \implies Q$.

Implication and English.

$P \implies Q$
Poll.

- If P, then Q.
- Q if P.
  Just reversing the order.
- P only if Q.
  Remember if P is true then Q must be true.
  this suggests that P can only be true if Q is true.
  since if Q is false P must have been false.
- P is sufficient for Q.
  This means that proving P allows you
  to conclude that Q is true.
  Example: Showing $n > 4$ is sufficient for showing $n > 3$.
- Q is necessary for P.
  For P to be true it is necessary that Q is true.
  Or if Q is false then we know that P is false.
  Example: It is necessary that $n > 3$ for $n > 4$.

Truth Table: implication.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg P \vee Q \quad \equiv P \implies Q$.

These two propositional forms are logically equivalent!

Contrapositive, Converse

- **Contrapositive** of $P \implies Q$ is $\neg Q \implies \neg P$.
  - If the plant pollutes, fish die.
  - If the fish don't die, the plant does not pollute.
    (contrapositive)
  - If you stand in the rain, you get wet.
  - If you did not stand in the rain, you did not get wet.
    (not contrapositive!) converse!
  - If you did not get wet, you did not stand in the rain.
    (contrapositive.)

  Logically equivalent! Notation: $\equiv$.
  $P \implies Q \equiv \neg P \vee Q \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \implies \neg P$.
- **Converse** of $P \implies Q$ is $Q \implies P$.
  If fish die the plant pollutes.
  Not logically equivalent!
- **Definition:** If $P \implies Q$ and $Q \implies P$ is P if and only if Q or $P \iff Q$.
  (Logically Equivalent: $\iff$.)

Variables.

Propositions?

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- $x > 2$
- n is even and the sum of two primes

No. They have a free variable.

We call them **predicates**, e.g., $Q(x) =$ "x is even"
Same as boolean valued functions from 61A!

- $P(n) =$ "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$."
- $R(x) =$ "$x > 2$"
- $G(n) =$ "n is even and the sum of two primes"
- Remember Wason's experiment!
  $F(x) =$ "Person x flew."
  $C(x) =$ "Person x went to Chicago
- $C(x) \implies F(x)$. Theory from Wason's.
  If person x goes to Chicago then person x flew.

Next: Statements about boolean valued functions!!

Quantifiers..

There exists quantifier:

$(\exists x \in S)(P(x))$ means "There exists an x in S where $P(x)$ is true."

For example:

$$(\exists x \in \mathbb{N})(x = x^2)$$

Equivalent to "$(0 = 0) \vee (1 = 1) \vee (2 = 4) \vee \ldots$"

Much shorter to use a quantifier!

For all quantifier;
$(\forall x \in S)\ (P(x))$. means "For all x in S, $P(x)$ is True ."

Examples:

"Adding 1 makes a bigger number."

$(\forall x \in \mathbb{N})\ (x + 1 > x)$

"the square of a number is always non-negative"

$(\forall x \in \mathbb{N})(x^2 >= 0)$

Wait! What is $\mathbb{N}$?

Quantifiers: universes.

Proposition: "For all natural numbers n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$."

Proposition has **universe**: "the natural numbers".

Universe examples include..

- $\mathbb{N} = \{0, 1, \ldots\}$ (natural numbers).
- $\mathbb{Z} = \{\ldots, -1, 0, \ldots\}$ (integers)
- $\mathbb{Z}^+$ (positive integers)
- $\mathbb{R}$ (real numbers)
- Any set: $S = \{Alice, Bob, Charlie, Donna\}$.
- See note 0 for more!

Other proposition notation(for discussion):
"$d|n$" means d divides n
or $\exists k \in \mathbb{Z}, n = kd$.
2|4? True.
4|2? False.

Back to: Wason's experiment:1

Theory: "If a person travels to Chicago, he/she flies."

Alice to Baltimore, Bob drove, Charlie to Chicago, and Donna flew.

Which cards do you need to flip to test the theory?

$Chicago(x)$ = "x went to Chicago." $Flew(x)$ = "x flew"

Statement/theory: $\forall x \in \{A, B, C, D\}, Chicago(x) \implies Flew(x)$

$Chicago(A)$ = False . Do we care about $Flew(A)$?
No. $Chicago(A) \implies Flew(A)$ is true.
since $Chicago(A)$ is False ,

$Flew(B)$ = False . Do we care about $Chicago(B)$?
Yes. $Chicago(B) \implies Flew(B) \equiv \neg Flew(B) \implies \neg Chicago(B)$.
So $Chicago(Bob)$ must be False .

$Chicago(C)$ = True . Do we care about $Flew(C)$?
Yes. $Chicago(C) \implies Flew(C)$ means $Flew(C)$ must be true.

$Flew(D)$ = True . Do we care about $Chicago(D)$?
No. $Chicago(D) \implies Flew(D)$ is true when $Flew(D)$ is true.

Only have to turn over cards for Bob and Charlie.

More for all quantifiers examples.

- "doubling a number always makes it larger"

  $(\forall x \in N)\ (2x > x)$ **False** **Consider** $x = 0$

  Can fix statement...

  $(\forall x \in N)\ (2x \geq x)$ **True**

- "Square of any natural number greater than 5 is greater than 25."

  $(\forall x \in N)(x > 5 \implies x^2 > 25)$.

  Idea alert: Restrict domain using implication.

Later we may omit universe if clear from context.

Quantifiers..not commutative.

- In English: "there is a natural number that is the square of every natural number".

  $(\exists y \in N)\ (\forall x \in N)\ (y = x^2)$ False

- In English: "the square of every natural number is a natural number."

  $(\forall x \in N)(\exists y \in N)\ (y = x^2)$ True

Quantifiers....negation...DeMorgan again.

Consider

$$\neg(\forall x \in S)(P(x)),$$

English: there is an x in S where $P(x)$ does not hold.

That is,

$$\neg(\forall x \in S)(P(x)) \iff \exists(x \in S)(\neg P(x)).$$

What we do in this course! We consider claims.

Claim: $(\forall x)\ P(x)$ "For all inputs x the program works."
For False , find x, where $\neg P(x)$.
Counterexample.
Bad input.
Case that illustrates bug.
For True : prove claim. Next lectures...

Negation of exists.

Consider

$$\neg(\exists x \in S)(P(x))$$

English: means that there is no $x \in S$ where $P(x)$ is true. English: means that for all $x \in S$, $P(x)$ does not hold.

That is,

$$\neg(\exists x \in S)(P(x)) \iff \forall(x \in S)\neg P(x).$$

Which Theorem?

Theorem: $(\forall n \in N)\ \neg(\exists a,b,c \in N)\ (n \geq 3 \implies a^n + b^n = c^n)$

Which Theorem?

Fermat's Last Theorem!

Remember Special Triangles: for $n = 2$, we have 3,4,5 and 5,7, 12 and ...

1637: Proof doesn't fit in the margins.

1993: Wiles ...(based in part on Ribet's Theorem)

DeMorgan Restatement:
Theorem: $\neg(\exists n \in \mathbb{N})\ (\exists a,b,c \in \mathbb{N})\ (n \geq 3 \implies a^n + b^n = c^n)$

Summary.

Propositions are statements that are true or false.

Proprositional forms use $\wedge, \vee, \neg$.

Propositional forms correspond to truth tables.

Logical equivalence of forms means same truth tables.

Implication: $P \implies Q \iff \neg P \vee Q$.

Contrapositive: $\neg Q \implies \neg P$
Converse: $Q \implies P$

Predicates: Statements with "free" variables.

Quantifiers: $\forall x\ P(x)$, $\exists y\ Q(y)$

Now can state theorems! And disprove false ones!

DeMorgans Laws: "Flip and Distribute negation"
$\neg(P \vee Q) \iff (\neg P \wedge \neg Q)$
$\neg\forall x\ P(x) \iff \exists x\ \neg P(x)$.

Next Time: proofs!